Invitation to Complex Analysis

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CHAPTER 4

Harmonic Functions and Conformal Mapping

19 *Harmonic Functions*

19A. Definitions

To say that a function $u(x,y)$ is **harmonic** in a region D means that u satisfies Laplace's equation,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

which is more conveniently written in subscript notation as $u_{11} + u_{22} = 0$. It is customary to consider only real-valued harmonic functions.

Eventually we shall be able to show (Sec. **20E**) that if a function u is continuous in a region and has just enough differentiability that u can and does satisfy Laplace's equation, then u actually has continuous partial derivatives of all orders. For the present, however, we adopt the working definition that a harmonic function has continuous first-order and second-order partial derivatives and satisfies Laplace's equation in a region.

The connection between harmonic functions and analytic functions is that both the real part u and the imaginary part v of an analytic function f satisfy Laplace's equation. This property follows from differentiating the Cauchy–Riemann equations (using that analytic functions have continuous second-order derivatives, as we have known since Sec. **7C**):

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y},$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}.$$

Since the two mixed second-order derivatives are equal, the function u is harmonic. Similarly (or because v is the real part of $-if$), the function v is harmonic.

Since one always cherishes the hope that sufficient conditions will turn out to be necessary, we might hope that, conversely, every harmonic function is the real part of some analytic function. This equivalence is not quite true. For example, if $u(x,y) = \ln(x^2+y^2)$, then u is harmonic in the annulus where $1 < |z| < 2$, but there is no (single-valued) analytic function in this annulus whose real part is u. Indeed, $u(x,y)$ is the real part of $\log z^2$, which cannot be defined even as a continuous function in the whole annulus.

In a simply connected region, however, a harmonic function u is indeed the real part of a function f that is analytic in the region.

We can (and shall in the next section) prove this proposition by actually constructing such an f. The imaginary part of f is called a **harmonic conjugate** of u ["a," not "the," because if c is any real number, then $f(z)+ic$ has the same real part as f]. The word "conjugate" here is supposed to convey the meaning of "associated," as does the same word in "complex conjugate," although the kind of association is quite different in the two phrases.

Because harmonic functions are connected with analytic functions in the way just described, theorems that we have proved about real parts of analytic functions (Exercises 16.2 and 16.4, for example, and the analog of Liouville's theorem in Sec. **16C**) are really theorems about harmonic functions.

The term "harmonic" is not restricted to two dimensions (although we shall not be concerned with harmonic functions in space): a solution of Laplace's equation in any number of dimensions is called harmonic. The theory of analytic functions in higher dimensions does not, however, correspond to harmonic functions in the same way as in two dimensions. Consequently, the study of Laplace's equation (and of the physical problems that it models) is more difficult in higher dimensions. In one dimension, on the other hand, the harmonic functions are just the affine functions $ax+b$, about which there is little to say.

19B. Finding a harmonic conjugate

Suppose now that we have a harmonic function u in a simply connected region D. To look for a harmonic function v such that $u+iv$ is analytic in D, we start from the Cauchy–Riemann equations: $u_1 = v_2$ and $u_2 = -v_1$. The first equation suggests that we ought to be able to find v by integrating u_1 with respect to y, since v_2 $(=u_1)$ is obtained by differentiating v with respect to y. This approach will work locally, but not necessarily over the whole of D, because if we start from a given point (x_0,y_0) of D, we cannot necessarily reach all points of D by moving just in the y direction. How to proceed? We could construct v, and hence our analytic function, in a neighborhood of (x_0,y_0) in this way and hope to extend the function to all of D by analytic continuation; we shall see that this approach works well for some specific functions. Alternatively, we ought to be able to find v by integrating $-u_2$ with respect to x, but this approach runs into the same difficulty as integrating with respect to y. Having progressed this far, we might have the idea to combine the two integration methods and to define v globally by a line integral:

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} [u_1(s,t)\,dt - u_2(s,t)\,ds],$$

where the integral is taken over an arc in D from (x_0,y_0) to (x,y). If this formula is to define a function v, then the integral needs to be independent of the path along which we integrate. Now Green's theorem shows that the integral is independent of the path (in a simply connected region) precisely when u satisfies Laplace's equation.

EXERCISE 19.1 Calculate the partial derivatives of v and so show that u and v are indeed the real part and the imaginary part of an analytic function in D.

19C. Formulas

Sometimes we are given a formula for a harmonic function, and we want to find a harmonic conjugate, or the associated analytic function, explicitly. If we have u and want v, then it is often convenient to do the integration locally and in stages. Integrating u_1 "partially" with respect to y gives a function that, after the addition of some function $\varphi(x)$, will be equal to $v(x,y)$. Now differentiate with respect to x to get $v_1(x,y)$, which is also equal to $-u_2(x,y)$. We can now find $\varphi(x)$ up to an additive constant, and hence obtain a formula for v. An example will clarify the procedure. Suppose $u(x,y) = x^2 - y^2$. Then $u_1 = 2x$ and $\int u_1\,dy = 2xy + \varphi(x) = v(x,y)$, whence $v_1(x,y) = 2y + \varphi'(x) = -u_2(x,y) = 2y$. Hence $\varphi'(x) = 0$, so φ is a constant, and $v(x,y) = 2xy + c$.

EXERCISE 19.2 Find conjugate harmonic functions of

(*a*) $x^3 - 3xy^2$,	(*b*) $e^{-y}\cos x$,
(*c*) $\log(x^2+y^2)$,	(*d*) $y/[(1-x)^2+y^2]$.

Often we are less interested in finding the harmonic conjugate function $v(x,y)$ than in finding the analytic function $f(z)$ explicitly in terms of z. In our example where $u(x,y) = x^2 - y^2$, it is easy to guess that $f(z) = z^2 + c$, but in more complicated cases, it is not always easy to express $u(x,y) + iv(x,y)$ as $f(z)$ even when we have explicit formulas for u and v. There are several short-cut methods for finding $f(z)$, and I list them here for reference. Caution: If you start from a function u that is not harmonic and use the general procedure, you will be stopped automatically at some stage, but *short-cuts can produce spurious results if you do not start from a harmonic u.*

There are three special methods for finding $f(z)$. At first sight, they all look fishy, but they are really all right, as we shall show after describing and illustrating them. You may prefer to skip ahead and read the proofs first before trying out the methods.[1]

RULE A *If you know the function u, and it is harmonic in some disk that contains an interval I of the real axis, define f' as follows:*

$$f'(z) = u_1(z,0) - iu_2(z,0).$$

Then integrate f' to get f in a neighborhood of I, and extend f by analytic continuation.

Note that the indicated partial derivatives have to be computed *before* replacing (x,y) by $(z,0)$.

Of course f is not uniquely determined by u: one can add to f a purely imaginary constant.

RULE B *If u is harmonic in a neighborhood of a point z_0 $(= x_0 + iy_0)$, then in this neighborhood we have, up to an additive imaginary constant,*

$$f(z) = 2u\left(\frac{z+\bar{z}_0}{2}, \frac{z-\bar{z}_0}{2i}\right) - u(x_0,y_0).$$

We may take z_0 equal to 0 if u is harmonic in a neighborhood of 0.

RULE C *If you know both u and v, and you know that they are harmonic conjugates in a neighborhood of 0, then*

$$f(z) = u(z,0) + iv(z,0).$$

These rules look outrageous at first sight, because we do not know (yet) that it makes any sense to replace the real variables x and y by complex numbers. Let us begin, however, by giving some illustrations to indicate that the rules are useful enough to justify some effort in proving that they really work.

Example 1 $u(x,y) = x^2 - y^2$.

Here $u_1 = 2x$, $u_2 = -2y$, and Rule A gives

$$f'(z) = 2z, \qquad \text{so} \qquad f(z) = z^2 + \text{constant}.$$

By Rule B, with $z_0 = 0$,

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) = 2\left(\frac{z^2}{4} + \frac{z^2}{4}\right) = z^2.$$

For Rule C, we are supposed to know that $v(x,y) = 2xy$; then

$$f(z) = (z^2 - 0) + i0 = z^2.$$

Example 2 $u(x,y) = e^{-y}\sin x$.

By Rule A,

$$f'(z) = \cos z + i\sin z = e^{iz}, \qquad \text{so} \qquad f(z) = -ie^{iz} + \text{constant}.$$

By Rule B, with $z_0 = 0$,

$$\begin{aligned} f(z) &= 2\exp\left(\frac{-z}{2i}\right)\sin\left(\frac{z}{2}\right) \\ &= \frac{1}{i}\exp\left(\frac{iz}{2}\right)(e^{iz/2} - e^{-iz/2}) \\ &= -ie^{iz} + i. \end{aligned}$$

By Rule C, if we know that $v = -e^{-y}\cos x$, then $f(z) = \sin z + i(-\cos z) = -ie^{iz}$.

Example 3 $u = \dfrac{x}{x^2+y^2}, \qquad u_1 = \dfrac{y^2-x^2}{(x^2+y^2)^2}, \qquad u_2 = \dfrac{-2xy}{(x^2+y^2)^2}.$

By Rule A,

$$f'(z) = -\frac{z^2}{z^4} = -\frac{1}{z^2}, \qquad \text{so} \qquad f(z) = \frac{1}{z} + \text{constant}.$$

By Rule B,

$$\begin{aligned} f(z) &= 2\frac{(z+\bar{z}_0)/2}{[(z+\bar{z}_0)/2]^2 + [(z-\bar{z}_0)/(2i)]^2} - \frac{x_0}{x_0^2+y_0^2} \\ &= \frac{z+\bar{z}_0}{z\bar{z}_0} - \frac{x_0}{x_0^2+y_0^2} = \frac{1}{z} + i\frac{y_0}{x_0^2+y_0^2}. \end{aligned}$$

Rule C is not applicable as it stands, because u is not defined at 0, but see the following exercise.

EXERCISE 19.3 Adapt Rule C to the case where u and v are harmonic conjugates in a neighborhood of a nonzero point z_0.

EXERCISE 19.4 Find analytic functions whose real parts are the functions in Exercise 19.2 and also $(x^2+y^2)^{1/4}\cos[\frac{1}{2}\tan^{-1}(y/x)]$.

Supplementary exercises

Find analytic functions with the following real parts.

1. $\tan^{-1}\left(\dfrac{y}{x}\right)$

2. $x^3y - xy^3$